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The Smirnov compactification functor is one-to-one over the class of complete first countable spaces

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Abstract

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It is proved that the Smirnov compactification functor σ: PROX $\rightarrow$ COMP (which relates to any proximity space (X, δ) its compactification σX corresponding to proximity δ) is one-to-one in the subcategory of complete first countable proximity spaces.

Keywords. Complete proximity (uniform, metric) space, compactification, Smirnov's compactification.

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Constructed by Stone and Čech in 1937 (see [1, 6]), the maximal bicompact extension βX of completely regular space X has a brilliant property: although in general the Stone-Čech functor is not one-to-one, over the class of first countable spaces it is [1]. Later on it was found that the Stone-Čech functor is a particular case of the Smirnov compactification functor, which relates to any proximity space (X, δ) its compactification σX corresponding to proximity δ [4]. But the Smirnov functor is one-to-one neither over the class of first countable spaces nor even over the class of metric spaces (e.g. if $X = (0, 1)$, $Y = (0, 1]$, $Z = [0, 1]$, then $\sigma X = \sigma Y = \sigma Z = Z$).

In this paper we show that, if one restricts oneself to *complete* spaces, the homeomorphness of Smirnov's compactifications σX and σY implies not only homeomorphness but even uniform homeomorphness of X and Y. The closure of a set S is denoted $[S]$.

Theorem 1. *Complete metric spaces with homeomorphic Smirnov's compactifications are uniformly equivalent.*

Proof. This is based on the following lemma:

Lemma 2. *In the remainder $\sigma X \backslash X$ of Smirnov's compactification of complete metric space (X, d), there exists no point of countable character.*

Proof. Assuming the opposite, let $x \in \sigma X \backslash X$ and $\{O_n\}$ be a countable base of open neighbourhoods of x. One can assume that $O_n \supset [O_{n+1}] \supset O_{n+1}$ $\forall n$ and $\operatorname{diam}(X, d) < \infty$. Then $F_n = [O_n] \cap X$ is a decreasing sequence of closed subsets of X, whose diameters decrease monotonically, but are not convergent to zero (otherwise, by virtue of completeness of X, the intersection of all F_n is not empty, but $\bigcap F_n = (\bigcap [O_n]) \cap X = \{x\} \cap X = \emptyset$). So there exist $\varepsilon > 0$ and a sequence of pairs of points a_n, $b_n \in F_n$ such that $d(a_n, b_n) > \varepsilon$. By Efremovich's lemma [2] we can choose a subsequence a_{n_k}, b_{n_k} such that the sets $A = \{a_{n_k} \mid k \in \mathbb{N}\}$ and $B = \{b_{n_k} \mid k \in \mathbb{N}\}$ are remote: $d(A, B) > \varepsilon/4$. But σX is the compactification of X corresponding to metric proximity, $[A] \ni x$, $[B] \ni x$, so $[A] \cap [B] \neq \emptyset$, hence A and B are close. The lemma is proved. □

Homeomorphism $f: \sigma X \to \sigma Y$ maps X onto Y and it is an equimorphism (=proximity isomorphism) between compacta σX and σY with respect to their unique proximities prolongating the metric proximities of X and Y. And for metric spaces equimorphness is equivalent to uniform isomorphness [2]. □

Corollary 3. *If metric spaces have homeomorphic Smirnov's compactifications, their completions are uniformly isomorphic.*

Proof. By [5, Theorem 7, p. 287], the completion $\tilde{X}$ of a metric space X lies in σX and consists precisely of all first countable points of this compactum. □

Note. Lemma 2 follows from [5, Theorem 6, p. 286]: every closed G_δ-set lying in the remainder $\sigma X \backslash X$ of complete metric space X has a potency of no less than the continuum. We have given its proof firstly because it is somewhat simpler than the proof of Smirnov's theorem (although the latter is stronger), and secondly, because we generalize it below to nonmetrizable spaces.

Let $(X, \mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ uniquely generates in X the proximity $\delta(\mathcal{U})$. The compactification of X corresponding to the proximity $\delta(\mathcal{U})$ is called the Samuel compactification.

Theorem 1′. *Complete uniform spaces of countable character with homeomorphic Samuel's compactifications are equimorphic.*

Lemma 2′. *In the remainder of Samuel's compactification of a complete uniform space, there exists no point of countable character.*

Proof. Let x, O_n, F_n be the same as above. Then there exist a symmetric entourage $U \in \mathcal{U}$ and pairs of points $a_n, b_n \in F_n$ such that $(a_n, b_n) \notin U$. Let us prove the analogue of Efremovich's lemma.

Lemma 4. *Let $(X, \mathcal{U})$ be a uniform space and a sequence of pairs of points a_n, b_n satisfy the condition $(a_n, b_n) \notin U$ for some symmetrical $U \in \mathcal{U}$. Then there exists a subsequence a_{n_k}, b_{n_k} such that the sets $A = \{a_{n_k} \mid k \in \mathbb{N}\}$ and $B = \{b_{n_k} \mid k \in \mathbb{N}\}$ are remote.*

Proof. Let us take a symmetrical entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$. One can assume that $\forall n$ there exist a finite number of points b_{n_k} such that $(a_n, b_{n_k}) \in V$. Otherwise the lemma would be proved: the subsequence a_{n_k}, b_{n_k} is the desired one, since $(a_{n_k}, b_{n_m}) \notin V$. Indeed, $(a_{n_k}, b_{n_m}) \in V$ implies $(a_{n_k}, b_{n_k}) = (a_{n_k}, b_{n_m}) \circ (b_{n_m}, a_n) \circ (a_n, b_{n_k}) \in V \circ V \circ V \subset U$, which contradicts the assumption. Similarly, one can assume that $\forall n$ there exists a finite number of points a_{n_k} such that $(a_{n_k}, b_n) \in V$. In order to choose the required subsequence, let us denote a_1 as a_{n_1} and b_1 as b_{n_1} and delete all pairs (a_n, b_n) such that $(a_1, b_n) \in V$ or $(a_n, b_1) \in V$ (there exists only a finite number of such points). Let us denote the first remaining pair as a_{n_2}, b_{n_2}. Iterating to infinity, we obtain all we need. Indeed, if $k \neq m$, then $(a_{n_k}, b_{n_m}) \notin V$ by choice, and if $k = m$, then $(a_{n_k}, b_{n_k}) \notin V$ and even $\notin U$ by assumption. Hence $V(A) \cap B = V(B) \cap A = \emptyset$. So A and B are remote. □

The rest of the theorem is proved as above. □

Now let us point out the following fact, which is seldom used. The theory of completeness of uniform and proximity spaces, developed by Smirnov [5], is applicable not only to uniform but also to pseudouniform spaces, i.e. to such spaces whose uniform structure satisfies axioms C1 and C3 but not necessarily C2 [5, p. 421] (recall that axiom C2 states "whenever U and V are entourages, then so is $U \cap V$"). In the set of all uniformities which generate in X the given proximity δ, there always exists a minimal uniformity [4, p. 563]. The corresponding completion is maximal and coincides with σX [5]. The maximal uniform structure does not necessarily exist [3], but the maximal pseudouniform structure Σ_c exists always [5]. The corresponding completion is minimal and coincides with $\tilde{X}$ (the completion of a proximity space (X, δ)), [5, p. 434]. A proximity space (X, δ) is complete iff the maximal pseudouniform space (X, Σ_c) compatible with it is the same. This can be taken as a definition of completeness of a proximity space [5]. One can easily verify that the proofs of Theorem 1′ and Lemmas 2′ and 4 are faithful for pseudouniform spaces too and obtain the following:

Theorem 1″. *Complete proximity spaces of countable character with homeomorphic Smirnov's compactifications are equimorphic.*

Note. The completion $(\tilde{X}, \tilde{\mathscr{U}})$ of a first countable uniform space $(X, \mathscr{U})$ (and even the completion $(\tilde{X}, \tilde{\delta})$ of a first countable proximity space (X, δ) which is equal to the intersection of the completions $(\tilde{X}, \tilde{\mathscr{U}})$ of all uniform spaces $(X, \mathscr{U})$ compatible with δ) can contain points of uncountable character. For example, if $X = \{\alpha \mid \alpha < \omega_1\}$ is the set of all countable ordinals with the order topology and the only compatible proximity δ, then $(\tilde{X}, \tilde{\delta}) = \sigma X = \beta X = X \cup \{\omega_1\}$. Nevertheless, Theorem 1″ (but not Theorem 1′) implies the following:

Corollary 5. *If proximity spaces of countable character have homeomorphic Smirnov's compactifications, then their completions are equimorphic.*

Proof. Evidently $\sigma X = \sigma X'$ for every X' such that $X \subset X' \subset \sigma X$. So $\sigma \tilde{X} = \sigma X$. Let $X_0 = \{x \in \sigma X \mid \chi(x) \leq \aleph_0\}$ be the set of all first countable points of σX. Obviously $X \subset X_0$. By Lemma 2′ $(\sigma \tilde{X} \backslash \tilde{X}) \cap X_0 = (\sigma X \backslash \tilde{X}) \cap X_0 = \emptyset$. Thus $X \subset X_0 \subset \tilde{X}$. Hence $\tilde{X}_0 = \tilde{X}$. Any homeomorphism $f: \sigma X \to \sigma Y$ maps X_0 onto Y_0. Therefore, $\tilde{X}_0 = \tilde{Y}_0$ and $\tilde{X} = \tilde{Y}$. □

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